Section 2-A
Life Cycle Costing

FINANCIAL MATHEMATICS

1.1. Time value of Money

A given amount of money, if properly invested, will earn interest and thus grow in magnitude. There is also a cost associated with the use of borrowed money. Personal finance for most people involves both of these effects of the time value of money.

Most everyone has placed some of his personal resources in a savings account where the savings institution promises to pay his some rate of return at a given interest rate depending on how long the investor will guarantee that these funds may be used by the savings institutions. If one invests money in savings certificates which cannot be cashed for one or two years, a larger interest rate will usually be obtained.

On the other hand, one usually has to borrow money sometime in their lifetime. Depending on the purpose of the loan, the interest rate will very. Mortgages are one common form of personal lending. Mortgages are accused loans where default by the borrower will result in the lending institution initiating legal proceedings to recover the loss by taking possession of the property. Mortgage loans have a lower rate of interest than unsecured loans because the lending institution takes less of a risk in loosing their money.

1.2 Cash flow distribution - continues or discrete

It should be recognized that in some business the flow of money is considered to take place continuously, while some savings institutions advertise continues compounding of interest, only money held for discrete intervals of time, usually quarterly, will earn interest.

The result of continuous or discrete cash flow and continuous or discrete compounding of interest is illustrated in Figure 1.1.

ACTUAL CASH FLOW POMPOUNDING OF INTEREST

Continuous Continuous

Discrete Discrete

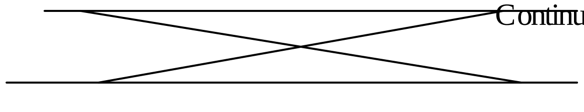

Fig. 1.1 Combination of Cash Flow and Interest

When stating the interest rates, the frequency of interest payment and the interest period must be define. In future discussions, it will be *assumed* that *the period is annual* and that *cash flow and interest are computed on a discrete annual basis.* The time period could be monthly just as well.

1.3 The Earning Power of Money

In business, funds borrowed for the prospect of gain are commonly exchanged for goods, services, or instruments of production. This leads to the consideration of the earning power of money that may make it profitable to borrow. Consider an example :

A man manually digs ditches for underground cable. He can dig 200 linear feet per day and receives $ 0.10 per linear foot Weather considerations limit his work to 180 days per year. Thus his maximum income would be:

Annual income = (200) (.1) (180) = $3,600

He has the alternative to purchase a power ditcher for $1,200 and he can borrow the money at 8% annual (discrete) interest. The machine can dig 800 linear feet per day and if he reduces his price to $0.06 per linear foot he can work every day that weather permits. At the end of the year the machine is worn out and has no scrap value.

Therefore, considering both receipts and disbursements

Receipts (money received by the business)

Amount of loan……………………………..$ 1,200
Payment for ditches dug
180 days x 800 ft/day x $0.06/ft………… $ 8,640
Total Received………….$ 9,840

Disbursements (money paid out by the business)

Purchase of ditcher…………………….$ 1,200
Fuel and Maintenance…………………$ 700
Interest on loan………………………..$ 96
Repayment of loan……………………$ 1,200
Total expenses………$ 3,196

Receipts less disbursements (Annual Income)…..$ 6,644

This is the amount received by the business. The man has now increased his earnings by $ 3,044.

Year	Amount at Beginning of Year	Interest Earned During year	Compound Amount at End of year
1.	1.00	.06	0.01 + .06 = 1.06
2.	1.06	.06	1.06 + .06 = 1.12
3.	1.12	.07	1.12 + .07 = 1.19
.			
7.	1.42	0.08	1.42 + .08 = 1.50

Table 1.1 Single Payment Compounding

The table illustrates the effect of single payment compounding. At the end of seven years the total amount is 150% of the initial amount. If invested at 6% interest, a given amount of money will double in 12 year; at 8 percent, it will double in 9 years; and at 16 percent, a sum would double in magnitude within 4 ½ years.

To generalize this concept, define terms where ;

P = Principal (amount invested now)
I = Annual interest (discrete)
N = Number of years

Then table 1.2 shows how the amount P will grow

Year	Amount at Beginning of Year	Interest Earned During Year	Compound Amount at the End of year
1.	P	Pi	P + Pi = P(1+i)
2.	P(1+I)	P(1+i)1	$P(1+i)+P(1+i)i = P(1+i)^2$
3.	$P(1+I)^2$	$P(1+i)^2I$	$P(1+i)^2+P(1+i)^2i = P(1+i)^3$
.			
N	$P(1+I)^{n-1}$	$P(1+i)^{n-1}I$	$(P1+i)^{n-1}+P(1+i)^{n-1}i = P(1+i)^n$
			= F

Table 1.2 Single Payment Compounding

The amount P is the present worth of today's investment while the amount F is the future value. This concept can be shown by a simple diagram, Figure 1.2

Fig. 1.3 Single Payment Compounding

Where :
(Single Payment) Present Value Factor $= \frac{1}{(1+i)^n}$
$= P/F\ i,n = V^n =$ P.V.F.

Where i and n were defined in section 1.4.1.

This means that $1.50 invested seven years from now, if interest is 6%, has a present worth of $1.00 now. Two thousand dollars, paid 12 years from now is worth on thousand dollars today if the interest rate is 6%.

Example

You need to have $1000 in a savings account five years from now. The bank is paying 7% and you want to know how much to invest no.

$P=F\ (P/F\ 7\%\ S_y) = \$1000\ (.71299) = \$\ 712.99$

The remainder of the formulas involve annuities which are equal amounts of money paid at equal intervals of time.

1.4.3 The Future Spot Cash Equivalent of an Annuity

Consider the amount accumulated if equal amounts A are invested at the end of each of n years. This would be typical of a retirement savings plan and is illustrated in figure 1.4.

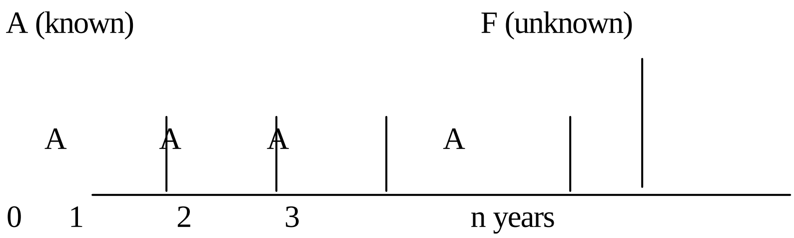

Fig. 1.4 Uniform Series Compounding

The total amount of all payments earning compound interest is

$$F = A(1+i)^{n-1}+A(1+i)^{n-2}...+A(1+i)+A \qquad (1.6)$$

The first payment A is multiplied by the compound interest factor for (n-1) years, the second payment for (n-2) years, and so forth until the last payment is made and at the same time the accumulated amount F is to be computed. Then

$$F(1+i) = A(1+i)^{n}+A(1+i)^{n-1}+\ldots+A(1+i)^{2}+A(1+i) \qquad (1.7)$$

Subtract first equation from the second,

$$Fi = A\,(1+i)^{n} = A \qquad (1.8)$$

(A/F 7%, 20yr)

$$A=F\wedge = \$1000 \quad \frac{.07}{(1+.07)^{20}-1}$$

$= \$1000\ (.02439) = \$\ 24.39$

The Value of (A/F) is taken from the tables in Appendix A.

1.4.5 <u>The Immediate Spot Cash Equivalent of a Future Annuity</u>

This is annuity is similar to the example of uniform series compounding where a given amount is saved annually but now the desired answer is the present worth of the accumulated amount. Consider Figure 1.6.

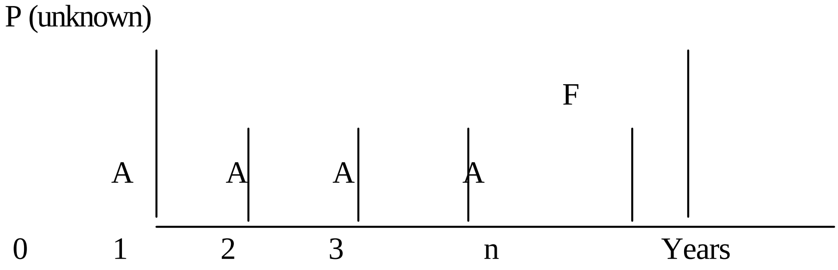

Fig. 1.6 Uniform Series Present Worth

The value of P is simply the present worth of F and F is related to A by equation 1.9.

$P = F\ (P/F\ i,n) = A(P/F\ i,n)\ (F/A\ i,n)$ (1.11)

$= A \left(\frac{1}{(1+i)^n}\right)\left(\frac{(1+i)^n - 1}{1}\right)$ (1.12)

$P = A\ \frac{(1+i)^n - 1}{i\,(1+i)^n} = A\ (P/A\ i,n)$ (1.13)

Uniform Series Present Worth Factor $= \frac{(1+i)^n - 1}{1(1+i)^n}$

$= P/A\ i,n = a = P.W.F.$

<u>Example</u>

What is the present worth of \$1 saved at 6% on December 31 of each year for 10 years?

P = \$1 (P/F 6%, 10 yr.)(F/A 6%, 10 yr.)
= \$1 (.55839) (13.18079)
P = \$1 (P/A 6%, 10 yr.)
= \$1 (7.36009)
P = \$ 7.36

1.4.6 The Future Annuity that is Equivalent to $1 Spot Cash

In this case the end result is known and the annual payment are not. Consider Figure 1.7.

Example

Money is borrowed today to buy a machine. The machine costs $100,000 and money costs 8%. The principle must be paid back in 20 years when the machine is worn out. How much must be act aside (and invested at 8%) annually?

A = P(A/P i.n) = 100,000 (.10185)

A = $10,185

1.4.7 Summary

The six important factors derived above are :

1. Compound Interest Factors $= (1+i)^n$ = F/P i,n = s^n = CIF
2. Present Value Factor $= \frac{1}{(1+i)^n}$ = P/F i,n = v^n = PVF
3. Compound Value Factor $= \frac{(1+i)^n-1}{1}$ = F/A i,n = S = CAF
4. Sinking Fund Factor $= \frac{i}{(1+i)^n-1}$ = A/F i,n = id = SFF
5. Present Worth Factor $= \frac{(1+i)^n-1}{I(1+i)^n}$ = P/A i,n = a = PWF
6. Capital Recovery Factor $= \frac{i(1+i)^n}{(1+i)^n-1}$ = A/P i,n = $\frac{1}{a}$ = CRF

1.5 Example Problem

The preceding formulas can be combined in various ways to solve financial problem. Two examples will be presented.

1.5.1. Example

A man retires at age 65 at which time he would like to have an annual pension of $10,00 for the next 30 years. He is 30 today and will make 35 equal payments starting one year from today. If he can invest his money at 5% what should be his annual payment ?

Solution

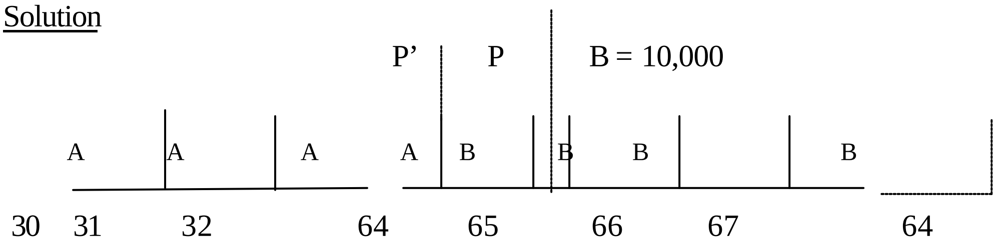

Fig. 1.7 Future worth of annuity